W35. Prove without any software:

$(\ln(e-1)+\ln(e-1))\ln\pi < \ln(\pi-1)+\ln(\pi+1)$

Daniel Sitaru

Solution by Arkady Alt, **San Jose**, **California**, **USA**.

Since for $x>1$ we have $\left(\frac{\ln(x-1)}{\ln x}\right)' = \frac{\frac{\ln x}{x-1}-\frac{\ln(x-1)}{x}}{(\ln x)^2} =$
$\frac{x\ln x-(x-1)(\ln(x-1))}{(\ln^2 x)x(x-1)} > 0$

then $\frac{\ln(x-1)}{\ln x}$ increase in $(1,\infty)$ and, therefore,

$e<\pi \Leftrightarrow e^2<\pi^2 \Rightarrow \frac{\ln(e^2-1)}{\ln e^2} < \frac{\ln(\pi^2-1)}{\ln\pi^2} \Leftrightarrow \frac{\ln(e^2-1)}{\ln e} < \frac{\ln(\pi^2-1)}{\ln\pi} \Leftrightarrow$
$(\ln(e-1)+\ln(e-1))\ln\pi < \ln(\pi-1)+\ln(\pi+1)$.

Remark. Elementary proof that $\frac{\ln(x-1)}{\ln x} = \log_x(x-1)$ increase in $(2,\infty)$.
For any $1<x$ and $h>0$ we have $\log_x\left(\frac{x}{x-1}\right) > \log_{x+h}\left(\frac{x}{x-1}\right)$ because for any $a>1$ function $\log_x a$ decrease by $x>1$. Since $\frac{x}{x-1} > \frac{x+h}{x+h-1}$
(because $\frac{x}{x-1}-\frac{x+h}{x+h-1} = \frac{h}{(x-1)(h+x-1)}$) then
$\log_{x+h}\left(\frac{x}{x-1}\right) > \log_{x+h}\left(\frac{x+h}{x+h-1}\right)$. Thus, we obtain
$\log_x\left(\frac{x}{x-1}\right) > \log_{x+h}\left(\frac{x+h}{x+h-1}\right) \Leftrightarrow -\log_x(x-1) > -\log_{x+h}(x+h-1) \Leftrightarrow$
$\log_{x+h}(x+h-1) > \log_x(x-1) \Leftrightarrow \frac{\ln(x+h-1)}{\ln(x+h)} > \frac{\ln(x-1)}{\ln x}$.